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Topological Vortices in the Modified Two-Dimensional Gross–Pitaevskii Theory

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It is shown that there exists an elementary vortex topological current in the modified two-dimensional Gross–Pitaevskii theory. An explicit expression for the vortex velocity field as a function of the order parameter is derived. The evolution of vortices is studied from the topological properties of the order parameter field. It is found that vortices generate or annihilate at the limit points and encounter, split, or merge at the bifurcation points of the order parameter field ψ.

KEY WORDS: topology; vortex; Gross–Pitaevskii theory.

1. INTRODUCTION

The observation of Bose–Einstein condensation (BEC) in trapped alkali vapors (Anderson *et al.*, 1995; Bradley *et al.*, 1995; Davis *et al.*, 1995) has intensified theoretical investigation on various aspects of the condensate. And it has been proven that the Gross–Pitaevskii (GP) mean-field theory (Dalfovo *et al.*, 1999; Gross, 1961; Pitaevskii, 1961) is an indispensable tool both in analyzing and predicting the outcome of experiments.

Recently, there has been a great deal of interest in low-dimensional BEC. Kolomeisky *et al.* have shown that the physics of dilute Bose systems requires a fundamental modification of the Gross–Pitaevskii theory in low dimensions $d \leq 2$ (Kolomeisky *et al.*, 2000). In particular, for the case of two dimensions (2D), the equation of motion for order parameter ψ is given by

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\delta E[\psi]}{\delta \psi^*}, \tag{1}$$

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where

$$E[\psi] = \frac{\hbar^2}{2m}\int d^2r\left[|\nabla\psi|^2 + \frac{2m}{\hbar^2}V(x)|\psi|^2 + \frac{4\pi^2}{|\ln(|\psi|^2a^2)|}|\psi|^4\right] \tag{2}$$

is the modified energy functional of the system. $V(x)$ is the external potential, a is the scattering length.

In this work we will show that, in the system described by Eqs. (1) and (2), there is an elementary vortex topological current constructed by the order parameter ψ. By means of the ϕ-mapping topological current theory, (Duan *et al.* 1998a,b; Duan and Meng, 1993; Yang, 1997) we obtain an explicit expression for the velocity of the vortex when the Jacobian determinant $D(\phi/x)$ at the zero points of the order parameter field ψ is nonzero, i.e., $D(\phi/x) \neq 0$. When this condition fails, i.e., $D(\phi/x) = 0$, we find that the branch processes of vortices happen at these points. This means that the vortex system is unstable there. According to the values of the vector Jacobian of the order parameter in $(2+1)$-dimensional space-time, the branch points are classified into two types: limit points and bifurcation points. Vortices are found generating or annihilating at the limit points and encountering, splitting, or merging at the bifurcation points of the order parameter field ψ.

This paper is organized as follows: In section 2, we construct a vortex topological current in the modified 2D GP theory and analyze its inner structure. In section 3, we concentrate on the branch processes of vortices in the theory. We present our concluding remarks in section 4.

2. VORTEX TOPOLOGICAL CURRENT IN THE MODIFIED 2D GROSS–PITAEVSKII THEORY

The order parameter ψ in the modified 2D GP theory can be regarded as the complex representation of a two-dimensional vector field $\vec{\phi} = (\phi^1, \phi^2)$ over the base space $R^2 \otimes R$ and

$$\psi = \phi^1 + i\phi^2. \tag{3}$$

Let us define the unit vector field n^a as

$$n^a = \frac{\phi^a}{\|\phi\|}, a = 1, 2, \tag{4}$$

satisfying

$$n^a n^a = 1.$$

We can construct a vortex topological current of the order parameter (Duan and Meng, 1993)

$$J^\mu = \frac{\hbar}{m}\epsilon^{\mu\nu\lambda}\epsilon_{ab}\partial_\nu n^a\partial_\lambda n^b, \mu, \nu, \lambda = 0, 1, 2, \tag{5}$$

and its time component is defined as the total density of the vortex charge: $J^0 = \rho$. Obviously, the current (5) is conserved,

$$\partial_\mu J^\mu = 0. \tag{6}$$

Using (4) and

$$\partial_\mu n^a = \frac{\partial_\mu \phi^a}{\|\phi\|} + \phi^a \partial_\mu \left(\frac{1}{\|\phi\|} \right), \tag{7}$$

J^μ changes into

$$J^\mu = \frac{\hbar}{m} \epsilon^{\mu\nu\lambda} \epsilon_{ab} \frac{\partial}{\partial \phi^c} \frac{\partial}{\partial \phi^a} (\ln \|\phi\|) \partial_\nu \phi^c \partial_\lambda \phi^b.$$

By defining vector Jacobians of ψ:

$$\epsilon^{ab} D^\mu \left(\frac{\phi}{x} \right) = \epsilon^{\mu\nu\lambda} \frac{\partial \phi^a}{\partial x^\nu} \frac{\partial \phi^b}{\partial x^\lambda}, \tag{8}$$

and making use of the Laplacian relation in ϕ space

$$\frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^a} (\ln \|\phi\|) = 2\pi \delta^2(\vec{\phi}),$$

we do have the δ-function-like topological current

$$J^\mu = \frac{h}{m} \delta^2(\vec{\phi}) D^\mu \left(\frac{\phi}{x} \right). \tag{9}$$

It is obvious that the current (9) is nonzero only when $\vec{\phi} = 0$, i.e.

$$\phi^a(x^0, x^1, x^2) = 0, \qquad a = 1, 2. \tag{10}$$

Suppose that there are N solutions of Eqs. (10), denoted by $z_l (l = 1, \ldots, N)$. According to the implicit function theorem, when the zero points z_l are regular points of $\vec{\phi}_x$ the Jacobian determinant must be nonzero:

$$D\left(\frac{\phi}{x} \right)|_{z_l} = D^0 \left(\frac{\phi}{x} \right)|_{z_l} \neq 0. \tag{11}$$

The solutions of Eqs. (10) can be generally obtained:

$$x^1 = z_l^1(t), \quad x^2 = z_l^2(t), \quad x^0 = t, \tag{12}$$

which represent the world lines of N vortices $\vec{z}_l (l = 1, \ldots, N)$. Using the implicit function theorem, we can get the velocity of the lth vortex from Eqs. (10),

$$\vec{v}_l = \frac{d\vec{z}_l}{dt} = \left[\frac{\vec{D}(\phi/x)}{D(\phi/x)} \right]_{\vec{z}_l}, \quad \vec{D}(\phi/x) = [D^1(\phi/x), D^2(\phi/x)]. \tag{13}$$

According to the δ-function theory and ϕ-mapping theory, (Duan *et al.*, 1998b) one can prove that

$$\delta^2(\vec{\phi}) = \sum_{l=1}^{N} \frac{\beta_l}{|D\left(\frac{\phi}{x}\right)|_{\vec{z}_l}} \delta^2(\vec{r} - \vec{z}_l(t)), \tag{14}$$

where β_l is the Hopf index of ϕ-mapping theory (Duan *et al.*, 1998b). The meaning of β_l is that when the point $\vec{r}$ covers the neighborhood of $\vec{z}_l$ once, the vector field $\vec{\phi}$ covers the corresponding region β_l times. Then the spatial and temporal components of the vortex three-current J^{μ}, $J^{1,2}$, and J^0, can be respectively written as the form of the current and the density of the system of N classical point particles with topological charge $\beta_l \eta_l h/m$ moving in the $(2+1)$-dimensional space-time

$$\vec{j} = \frac{h}{m} \sum_{l=1}^{N} \beta_l \eta_l \vec{v}_l \delta^2(\vec{r} - \vec{z}_l(t)), \tag{15}$$

$$\rho = \frac{h}{m} \delta^2(\vec{\phi}) D\left(\frac{\phi}{x}\right) = \frac{h}{m} \sum_{l=1}^{N} \beta_l \eta_l \delta^2(\vec{r} - \vec{z}_l(t)),$$

where $\eta_l = sgn(D(\phi/x)_{z_l}) = \pm 1$ is the Brouwer degree (Duan *et al.*, 1998b). Vortex corresponds to $\eta_l = +1$, while antivortex corresponds to $\eta_l = -1$. According to Eq. (6), the topological charge of vortex is conserved:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0. \tag{16}$$

The solutions (12) of Eqs. (10) are based on the condition that the Jacobian determinant $D^0(\phi/x) = D(\phi/x) \neq 0$. When $D^0(\phi/x) = 0$, i.e., η_l is indefinite, the above results (12) will change in some way. It is interesting to discuss what will happen and what the correspondence in physics is when this condition fails.

3. BRANCH PROCESSES OF VORTICES IN THE MODIFIED 2D GROSS–PITAEVSKII THEORY

When the usual Jacobian $D^0(\phi/x) = 0$ at some points, it is shown that there exist several crucial cases of branch process of vortices at these points, which are called branch points. There are two kinds of branch points, namely limit points and bifurcation points. Each kind corresponds to different cases of branch processes.

3.1. Generating and Annihilating of Vortices

First, we study the case that the zeroes of the order parameter field ψ include some limit points. The limit points are determined by Eqs. (10) and

$$D^0\left(\frac{\phi}{x}\right)|_{(t^*,\vec{z}_l)} = 0, \qquad D^1\left(\frac{\phi}{x}\right)|_{(t^*,\vec{z}_l)} \neq 0, \tag{17}$$

or

$$D^0\left(\frac{\phi}{x}\right)|_{(t^*,\vec{z}_l)} = 0, \qquad D^2\left(\frac{\phi}{x}\right)|_{(t^*,\vec{z}_l)} \neq 0. \tag{18}$$

For simplicity, we only consider the case (17) and denote one of the limit points as $(t^*, \vec{z}_l)$. Taking account of (17) and using the implicit function theorem, we have a unique solution of Eqs. (10) in the neighborhood of the point $(t^*, \vec{z}_l)$

$$t = t(x^1), \qquad x^2 = x^2(x^1) \tag{19}$$

with $t^* = t(z_l^1)$. In this case, one can see that

$$\frac{dx^1}{dt}|_{(t^*,\vec{z}_l)} = \frac{D^1(\phi/x)}{D(\phi/x)}|_{(t^*,\vec{z}_l)} = \infty, \tag{20}$$

i.e.,

$$\frac{dt}{dx^1}|_{(t^*,\vec{z}_l)} = 0. \tag{21}$$

The Taylor expansion of $t = t(x^1)$ at the limit points $(t^*, \vec{z}_l)$ is (Duan *et al.*, 1998a)

$$t - t^* = \frac{1}{2}\frac{d^2t}{(dx^1)^2}|_{(t^*,\vec{z}_l)}\left(x^1 - z_l^1\right)^2, \tag{22}$$

which is a parabola in the $x^1 - t$ plane. From Eq. (22) we can obtain two solutions $x_1^1(t)$ and $x_2^1(t)$, which give two branch solutions (world lines of vortices). If

$$\frac{d^2t}{(dx^1)^2}|_{(t^*,\vec{z}_l)} > 0,$$

we have the branch solutions for $t > t^*$ [see Fig. 1(a)]; otherwise, we have the branch solutions for $t < t^*$ [see Fig. 1(b)]. These two cases are related to the orgin and annihilation of a vortex–antivortex pair.

One of the results of Eq. (20), that the velocity of vortices is infinite when they are annihilating, agrees with the fact obtained by Bray (1997), who has a scaling argument associated with the point defects final annihilation which leads to a large velocity tail. From Eq. (20), we also obtain a new result that the velocity of vortices is infinite when they are generating, which is gained only from the topology of the order parameter field.

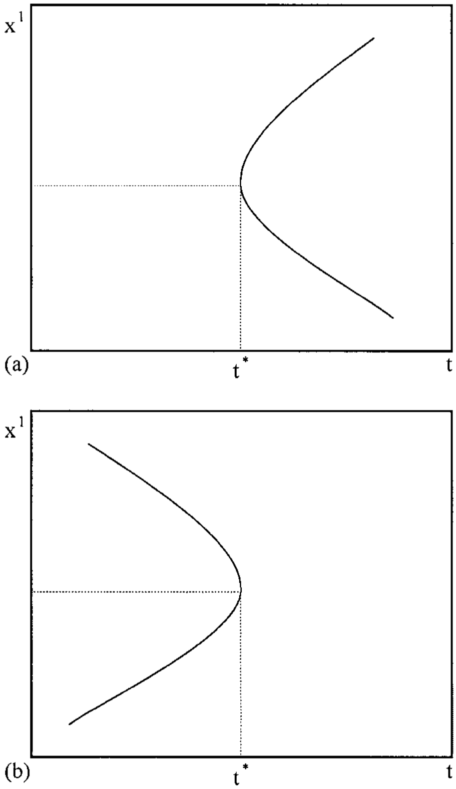

Fig. 1. Projecting the world lines of vortices onto $(x^1 - t)$-plane. (a) The branch solutions for Eq. (22) when $d^2t/(dx^1)^2|_{(t^*,\vec{z}_l)} > 0$, i.e., a vortex–antivortex pair generates at the limit point. (b) The branch solutions for Eq. (22) when $d^2t/(dx^1)^2|_{(t^*,\vec{z}_l)} < 0$, i.e., a vortex–antivortex pair annihilates at the limit point.

Since the topological current is identically conserved, the topological charges of these two generated or annihilated vortices must be opposite at the limit points, i.e.,

$$\beta_1\eta_1 + \beta_2\eta_2 = 0. \tag{23}$$

For a limit point it is required that $D^1(\phi/x)|_{(t^*,\vec{z}_l)} \neq 0$. As to a bifurcation point, it must satisfy a more complex condition (Kubicek and Marek, 1983). This case will be discussed in the following section.

3.2. Encountering, Splitting, and Merging of Vortices

Let us turn to the case in which the restrictions of Eqs. (10) are

$$D^i\left(\frac{\phi}{x}\right)\Big|_{(t^*,\vec{z}_l)} = 0, \qquad i = 0, 1, 2. \tag{24}$$

These three restrictive conditions will lead to an important fact that the functional relationship between t and x^1 or x^2 is not unique in the neighborhood of $(t^*, \vec{z}_i)$. This fact is easily seen from

$$\frac{dx^1}{dt} = \frac{D^1(\phi/x)}{D^0(\phi/x)}\Big|_{(t^*,\vec{z}_l)}, \qquad \frac{dx^2}{dt} = \frac{D^2(\phi/x)}{D^0(\phi/x)}\Big|_{(t^*,\vec{z}_l)}, \tag{25}$$

which under (24) directly shows that the direction of the integral curve of (25) is indefinite at $(t^*, \vec{z}_l)$. Therefore, the very point $(t^*, \vec{z}_l)$ is called a bifurcation point of the order parameter. Assume that the bifurcation point $(t^*, \vec{z}_l)$ has been found from Eqs. (10) and (24). We know that, at the bifurcation point $(t^*, \vec{z}_l)$, the rank of the Jacobian matrix $[\partial\phi/\partial x]$ is 1. With the aim of finding the different directions of all branch curves of Eqs. (10) at the bifurcation point, we suppose that

$$\frac{\partial\phi^1}{\partial x^2}\Big|_{(t^*,\vec{z}_l)} \neq 0. \tag{26}$$

According to the ϕ-mapping theory, the Taylor expansion of the solution of Eqs. (10) in the neighborhood of the bifurcation point $(t^*, \vec{z}_l)$ can be expressed as (Duan *et al.*, 1998a)

$$A\left(x^1 - z_l^1\right)^2 + 2B\left(x^1 - z_l^1\right)(t - t^*) + C(t - t^*)^2 = 0, \tag{27}$$

which leads to

$$A\left(\frac{dx^1}{dt}\right)^2 + 2B\frac{dx^1}{dt} + C = 0 \tag{28}$$

and

$$C\left(\frac{dt}{dx^1}\right)^2 + 2B\frac{dt}{dx^1} + A = 0, \tag{29}$$

where A, B, and C are three constants. The solutions of Eq. (28) or Eq. (29) give different directions of the branch curves (world lines of vortices) at the bifurcation point. There are four possible cases, which will show the physical meanings of the bifurcation points.

Case 1 ($A \neq 0$). For $\Delta = 4(B^2 - AC) > 0$ from Eq. (28) we get two different directions of the velocity field of vortices

$$\frac{dx^1}{dt}\Big|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A}, \tag{30}$$

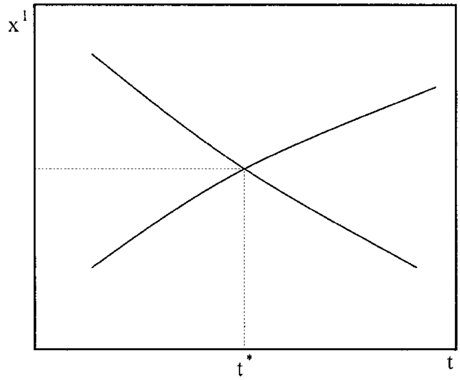

Fig. 2. Two world lines of vortices intersect with different directions at the bifurcation point, i.e., two vortices encounter at the bifurcation point.

which is shown in Fig. 2, where two world lines of two vortices intersect with different directions at the bifurcation point. This shows that two vortices encounter and then depart at the bifurcation point.

Case 2 ($A \neq 0$). For $\Delta = 4(B2 - AC) = 0$ from Eq. (28) we obtain only one direction of the velocity field of vortices

$$\frac{dx^1}{dt}|_{1,2} = -\frac{B}{A}, \tag{31}$$

which include three important cases: (a) Two world lines tangentially contact, i.e., two vortices tangentially encounter at the bifurcation point [see Fig. 3(a)]; (b) Two world lines merge into one world line, i.e., two vortices merge into one vortex at the bifurcation point [see Fig. 3(b)]; and (c) One world line resolves into two world lines, i.e., one vortex splits into two vortices at the bifurcation point [see Fig. 3(c)].

Case 3 ($A = 0, C \neq 0$). For $\Delta = 4(B^2 - AC) = 0$ from Eq. (29) we have

$$\frac{dt}{dx^1}|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{C} = 0, -\frac{2B}{C}. \tag{32}$$

There are two important cases: (a) One world line resolves into three world lines, i.e., one vortex splits into three vortices at the bifurcation point [see Fig. 4(a)]; and (b) Three world lines merge into one world line, i.e., three vortices merge into one vortex at the bifurcation point [see Fig. 4(b)].

Case 4 ($A = C = 0$). Eq. (28) and Eq. (29) give respectively

$$\frac{dx^1}{dt} = 0, \qquad \frac{dt}{dx^1} = 0. \tag{33}$$

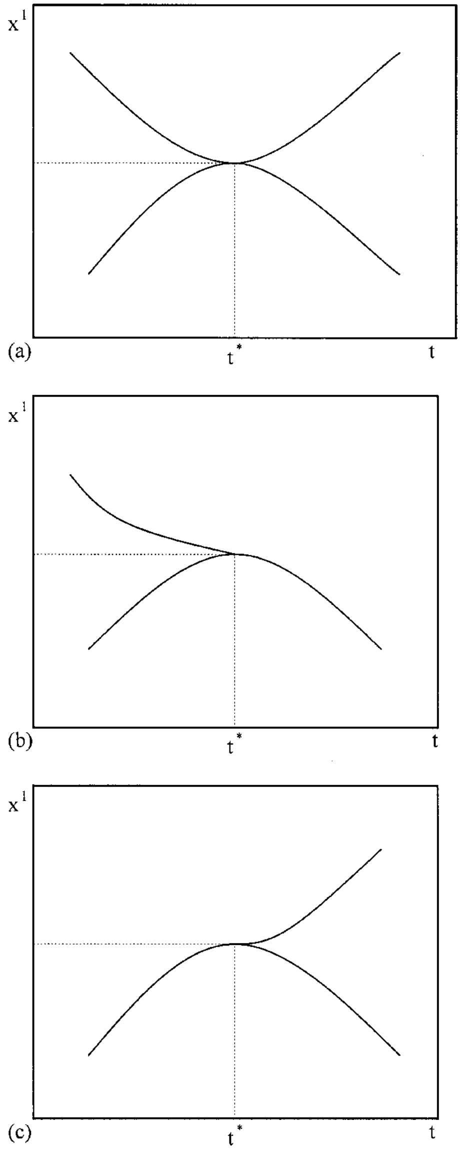

Fig. 3. (a) Two world lines tangentially contact, i.e., two vortices tangentially encounter at the bifurcation point. (b) Two world lines merge into one world line, i.e., two vortices merge into one vortex at the bifurcation point. (c) One world line resolves into two world lines, i.e., one vortex splits into two vortices at the bifurcation point.

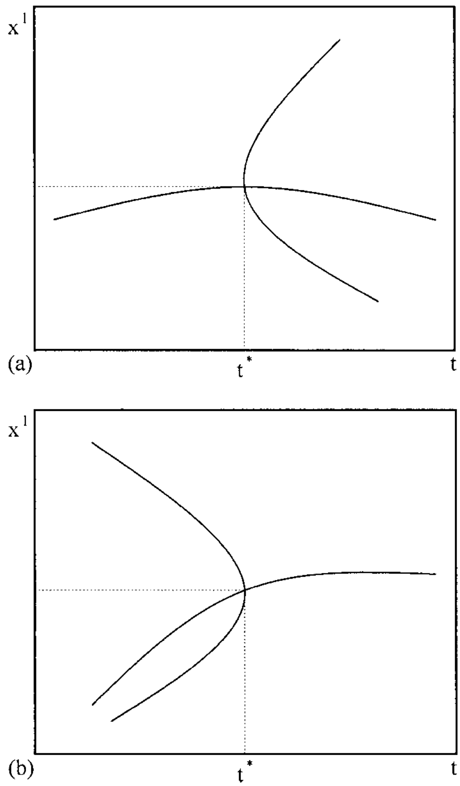

Fig. 4. Two important cases of Eq. (32). (a) One world line resolves into three world lines, i.e., one vortex splits into three vortices at the bifurcation point. (b) Three world lines merge into one world line, i.e., three vortices merge into one vortex at the bifurcation point.

This case is obvious, [see Fig. 5] and is similar to Case 3.

The above solutions reveal the evolution of the vortices. Besides the encountering of the vortices, i.e., two vortices encounter and then depart at the bifurcation point along different branch curves [see Fig. (2) and Fig. 3(a)], it also includes splitting and merging of vortices. When a multicharged vortex moves through the bifurcation point, it may split into several vortices along different branch curves [see Fig. 3(c), Fig. 4(a), and Fig. 5(b)]. On the contrary, several vortices can merge into one vortex at the bifurcation point [see Fig. 3(b) and Fig. 4(b)].

The indentical conversation of the topological charge shows the sum of the topological charge of these final vortices must be equal to that of the original

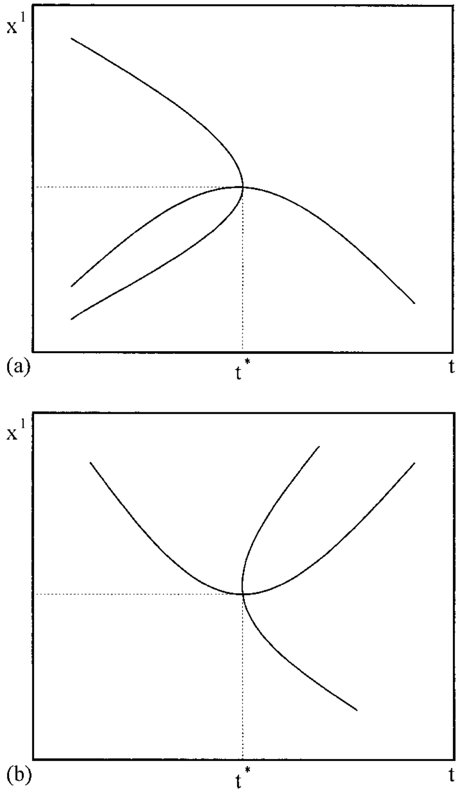

Fig. 5. Two world lines intersect normally at the bifurcation point. This case is similar to Fig. 4. (a) Three vortices merge into one vortex at the bifurcation point. (b) One vortex splits into three vortices at the bifurcation point.

vortices at the bifurcation point, i.e.,

$$\sum_i \beta_i \eta_i = \sum_f \beta_f \eta_f. \tag{34}$$

Furthermore, from the above studies, we see that the generation, annihilation, and bifurcation of vortices are not gradually changed, but suddenly changed at the critical points.

4. CONCLUSION

In this work we studied the inner topological structure of vortex system in the modified two-dimensional Gross–Pitaevskii theory in detail. One shows that the vortices are generated from $\phi = 0$ naturally in the framework of our topological current theory and the charge of vortex is characterized by Hopf index and Brouwer degree. One also shows that the quantities such as density and velocity of vortex expressed in terms of the order parameter can be rigorously determined by the ϕ-mapping theory. Futhermore, we conclude that there exist crucial cases of branch processes in the evolution of the vortices when $D(\phi/x) = 0$, i.e., η_l is indefinite. This means that the vortices generate or annihilate at the limit points and encounter, split, or merge at the bifurcation points of the order parameter field, which means that the vortex system is unstable at these branch points. It is shown that the phenomenological analysis of the topology of the order parameter gives a qualitative explanation of vortex state in the modified 2D GP theory, which is a necessary preliminary to microscopic considerations (of how vortices generate, annhilate, etc).

Here we must point out that there exist two restrictions of the evolution of vortices. One restriction is the conservation of the topological charge of the vortices during the branch process [see Eqs. (23) and (34)], the other restriction is the number of different directions of the world lines of vortices is at most four at a space-time point [see Eqs. (28) and (29)]. These two restrictions lead to some interesting results of the evolution of vortices in the modified 2D GP theory. For example, a multiply quantized vortex with $\beta_l > 3$ can never directly break up into unit vortices ($\beta_l = 1$). It is interesting to find a branch point and the concrete branch process in the neighborhood of it. We hope the results of this paper will help those experts to build simulations of vortex dynamics in the modified 2D GP theory.

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